Birzeit University
Faculty of Science-Department of Physics
Quantum Mechanics I Phys433
Fall 2018
2^{nd} Exam, December 16, 2018

1. (15 points) If $\hat{A}_s$ is an operator in the Shrödinger picture and $\hat{A_H}$ is the corresponding operator in the Heisenberg picture, Such that

$$\hat{A_H} = e^{\frac{-i\hat{H}t}{\hbar}} \hat{A}_s e^{\frac{i\hat{H}t}{\hbar}}$$

Under what conditions $\hat{A_H}$ can be time-independent. Justify your answer.
A_s must be time independent and commute with the Hamiltonian.

2. A Hamiltonian H has two orthonormal eigenstates $|1>$ and $|2>$ such that:

$$\hat{H}|1>= E_1|1> \qquad \hat{H}|2>= E_2|2> \qquad E_1 \neq E_2$$

Two states $|A>$ and $|B>$ are defined as follows:

$$<1|A>= \frac{1}{\sqrt{2}} \qquad <2|A>= \frac{i}{\sqrt{2}}$$
$$<1|B>= \frac{1}{\sqrt{2}} \qquad <2|B>= \frac{-i}{\sqrt{2}}$$

(a) (6 points) Calculate $<A|B>$ and $<B|A>$

$$\begin{aligned} <A|B> &= <A|I|B>=<A|(|1><1|+|2><2|)|B> \\ &= <A|1><1|B>+<A|2><2|B> \\ &= \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} + \frac{-i}{\sqrt{2}} \times \frac{-i}{\sqrt{2}} = 0 \end{aligned}$$

(b) (6 points) If the system initially in the state $|\psi(t=0)>= |A>$, what is the time dependent state $|\psi(t)>$?

$$|\psi(t=0)>= |A>= |1>+|2>$$
$$|\psi(t)>= |1> e^{-iE_1t/\hbar} + |2> e^{-iE_2t/\hbar}$$

(c) (8 points) What is the probability of finding the particle at state $|A>$ at any time t?
Probability is defined as:

$$|<A|\psi(t)>|^2 = |e^{-iE_1t/\hbar} + e^{-iE_2t/\hbar}|^2$$

3. Consider an electron in the Hydrogen-atom. The wavefunction of the electron is, at time t =0, written as:

$$\Psi(r, t = 0) = A(\psi_{211} + 2\psi_{300} + \psi_{421})$$

(a) (4 points) Find the normalization constant A

Since ψ_{nlm} forms an orthonormal set then:

$$\begin{aligned} 1 &= A^2(1 + 4 + 1) \\ A &= \frac{1}{\sqrt{6}} \end{aligned}$$

(b) (4 points) Write the wavefunction at any later time t

$$\Psi(r, t = 0) = \frac{1}{\sqrt{6}}(\psi_{211}e^{-iE_2t/\hbar} + 2\psi_{300}e^{-iE_3t/\hbar} + \psi_{421}e^{-iE_4t/\hbar})$$

(c) (4 points) What is the expectation value of L_z

Since ψ_{nlm} is an eigenvector of L_z, then

$$\begin{aligned} < L_z > &= \sum c_n^2 L_z \\ &= \frac{1}{6}(\hbar) + \frac{4}{6}(0) + \frac{1}{6}(\hbar) = \hbar/3 \end{aligned}$$

(d) (4 points) What is the expectation value of L^2

Since ψ_{nlm} is an eigenvector of L^2, then

$$\begin{aligned} < L^2 > &= \sum c_n^2 \hbar^2 l(l + 1) \\ &= \frac{1}{6}(2\hbar^2) + \frac{4}{6}(0\hbar^2) + \frac{1}{6}(6\hbar^2) = 4\hbar^2/3 \end{aligned}$$

(e) (4 points) What is the expectation value of H

Since ψ_{nlm} is an eigenvector of H, then

$$\begin{aligned} < H > &= \sum c_n^2 E_n \\ &= \frac{1}{6}(E_2) + \frac{4}{6}(E_3) + \frac{1}{6}(E_4) \end{aligned}$$

4. Consider a particle with spin-angular momentum j=3/2. There are four sub-levels with this value of j but different eigenvalues of $\hat{j}_z$. They are written in the following form $|jm_j>$: $|\frac{3}{2}\frac{3}{2}>$, $|\frac{3}{2}\frac{1}{2}>$, $|\frac{3}{2}\frac{-1}{2}>$, and $|\frac{3}{2}\frac{-3}{2}>$.

(a) (6 points) Show that the raising operator can be written in the following form:

$$\hat{j}_+ = \sqrt{3}|\frac{3}{2}\frac{3}{2}><\frac{3}{2}\frac{1}{2}| + 2|\frac{3}{2}\frac{1}{2}><\frac{3}{2}\frac{-1}{2}| + \sqrt{3}|\frac{3}{2}\frac{-1}{2}><\frac{3}{2}\frac{-3}{2}|$$

We know(can calculate) the following relations:

$$\begin{aligned}
j_+|\frac{3}{2}\frac{3}{2}> &= 0\\
j_+|\frac{3}{2}\frac{1}{2}> &= \sqrt{3}|\frac{3}{2}\frac{3}{2}>\\
j_+|\frac{3}{2}\frac{-1}{2}> &= 2|\frac{3}{2}\frac{1}{2}>\\
j_+|\frac{3}{2}\frac{-3}{2}> &= \sqrt{3}|\frac{3}{2}\frac{-1}{2}>\\
j_+ &= j_+(|\frac{3}{2}\frac{3}{2}><\frac{3}{2}\frac{3}{2}| + |\frac{3}{2}\frac{1}{2}><\frac{3}{2}\frac{1}{2}| + |\frac{3}{2}\frac{-1}{2}><\frac{3}{2}\frac{-1}{2}| + |\frac{3}{2}\frac{-3}{2}><\frac{3}{2}\frac{-3}{2}|)
\end{aligned}$$

and now we can get the required results.

(b) (3 points) Write $\hat{j}_-$

We know that $j_- = j_+^\dagger$

$$\hat{j}_+ = \sqrt{3}|\frac{3}{2}\frac{1}{2}><\frac{3}{2}\frac{3}{2}| + 2|\frac{3}{2}\frac{-1}{2}><\frac{3}{2}\frac{1}{2}| + \sqrt{3}|\frac{3}{2}\frac{-3}{2}><\frac{3}{2}\frac{-1}{2}|$$

(c) (8 points) Find the matrix representation of $\hat{j}_x$, $\hat{j}_y$, and $\hat{j}_z$

$$\hat{j}_z = \hbar\begin{pmatrix} 3/2 & 0 & 0 & 0\\ 0 & 1/2 & 0 & 0\\ 0 & 0 & -1/2 & 0\\ 0 & 0 & 0 & -3/2 \end{pmatrix}$$

$$\hat{j}_+ = \hbar\begin{pmatrix} 0 & \sqrt{3} & 0 & 0\\ 0 & 0 & 2 & 0\\ 0 & 0 & 0 & \sqrt{3}\\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\hat{j}_- = \hbar\begin{pmatrix} 0 & 0 & 0 & 0\\ \sqrt{3} & 0 & 0 & 0\\ 0 & 2 & 0 & 0\\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$$

$$\hat{j}_x = \frac{\hat{j}_+ + \hat{j}_-}{2} = \frac{\hbar}{2}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0\\ \sqrt{3} & 0 & 2 & 0\\ 0 & 2 & 0 & \sqrt{3}\\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$$

$$\hat{j}_x = \frac{\hat{j}_+ + \hat{j}_-}{2} = \frac{\hbar}{2}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0\\ \sqrt{3} & 0 & 2 & 0\\ 0 & 2 & 0 & \sqrt{3}\\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$$

(d) (4 points) let $|\psi(t=0)> = \frac{1}{2\sqrt{2}}(|\frac{3}{2}\frac{3}{2}> + \sqrt{3}|\frac{3}{2}\frac{1}{2}> + \sqrt{3}|\frac{3}{2}\frac{1}{2}> + |\frac{3}{2}\frac{-3}{2}>)$ Show that it is an eigenvector of j_x.

One can write the wavefunction in matrix form as:

$$\hat{j}_x == \frac{1}{2\sqrt{2}}\begin{pmatrix} 1\\ \sqrt{3}\\ \sqrt{3}\\ 1 \end{pmatrix}$$

Now one can multiply the matrix of $\hat{j}_x$ with the wave function and show that this wave function is indeed an eignevector $\hat{j}_x$ with an eigenvalue of 3/2

(e) (4 points) If the particle a magnetic moment $\vec{\mu} = g\vec{j}$, and is placed in uniform magnetic field that points in the x-direction. The particle initial wave-function is given in the previous part. Find $< \hat{j}_z > (t)$

One can notice that the Hamiltonian is probational to j_x and thus the wavefunction in the previous part is a stationary state. Thus the expectation value of any operator doesn't depend on time. $< j_z > (t) = 0$

5. (20 points) Show that any operator that commutes with two Cartesian components of the angular momentum operator necessarily commutes with the total angular momentum operator.

We are given the following:

$$[A, L_i] = [A, L_j] = 0$$

All what we need now is to show that

$$[A, L_k] = 0$$

We know that

$$[L_i, L_j] = i\hbar L_k$$

Now:

$$\begin{aligned}
[A, L_z] &= \left[A, \frac{i}{\hbar}[L_i, L_j]\right] \\
&= \frac{i}{\hbar}[A, L_i L_j - L_j L_i] \\
&= \frac{i}{\hbar}([A, L_i L_j] - [A, L_j L_i]) \\
&= 0
\end{aligned}$$

Question:	1	2	3	4	5	Total
Points:	15	20	20	25	20	100
Score:						

Good Luck